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SELECTED WORKS

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COMMENTS ON NICOD'S AXIOM AND ON "GENERALIZING DEDUCTION" *)

In the present paper I use the following bibliographical abbreviations:

"Ajdukiewicz" for "*Główne zasady metodologii nauk i logiki formalnej* (Fundamental principles of the methodology of science and of formal logic). Lectures delivered by Professor K. Ajdukiewicz at the University of Warsaw in the academic year 1927/1928. Authorized lecture notes edited by M. Presburger. Publications of the Association of Students of Mathematics and Physics of the University of Warsaw. Vol. XVI, 1928."

"Kotarbiński" for "Tadeusz Kotarbiński, *Elementy teorii poznania, logiki formalnej i metodologii nauk* (Elements of epistemology, formal logic, and the methodology of science). The Ossolineum Publishers, Lwów 1929."**)

"Leśniewski" for "Dr. Phil. Stanisław Leśniewski, a.o. Professor der Philosophie der Mathematik an der Universität Warszawa, *Grundzüge eines neuen Systems der Grundlagen der Mathematik*, Einleitung und §§ 1–11. Sonderabdruck (mit unveränderter Pagination) aus dem XIV. Bande der *Fundamenta Mathematicae*. Warsaw, 1929."

"Łukasiewicz (1)" for "Jan Łukasiewicz, *O znaczeniu i potrzebach logiki matematycznej* (On the significance and requirements of mathematical logic). *Nauka Polska*, Vol. X, Warsaw 1929."

"Łukasiewicz (2)" for "Dr. Jan Łukasiewicz, Professor of the University of Warsaw, *Elementy logiki matematycznej* (Elements of mathematical logic). Authorized lecture notes prepared by M. Presburger.

*) First published as "Uwagi o aksjomacie Nicoda i 'dedukcji uogólniającej' " in *Księga pamiątkowa Polskiego Towarzystwa Filozoficznego*, Lwów, 1931. Reprinted in the 1961 edition *Z zagadnień logiki i filozofii*.

**) Available in English under the title *Gnosiology* (published jointly in 1966 by Zakład Narodowy im. Ossolińskich and Pergamon Press).

Publications of the Association of Students of Mathematics and Physics of the University of Warsaw. Vol. XVIII, 1929."*)

The items "Ajdukiewicz" and "Łukasiewicz (2)" are lithographed.

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* *

At the beginning of his German-language treatise on the foundations of mathematics Dr. Leśniewski parenthetically mentions (with my approval) a certain "simplification" of Nicod's axiom, made by me in 1925 and consisting in the reduction of the different variables that occur in this axiom from five to four. [1]) Since I have not so far published this result of my research, I shall do it in this paper so that Dr. Leśniewski's reference, based only on a manuscript source, may have a foundation in a printed publication.

I do this the more willingly as I can at the same time settle another issue. In transforming Nicod's axiom I encountered for the first time a case of deductive inference in which the conclusion is more general than the premiss. The second part of the present paper is concerned with that "generalizing deduction", which may prove to be of interest not only to logicians, but to philosophers as well.

I

1. Nicod's axiom can, with the use of parentheses, be written in the following way: [2])

(K) $\{p/(q/r)\}/[\{t/(t/t)\}/\{(s/q)/((p/s)/(p/s))\}]$.

In the parenthesis-free symbolism this becomes: [3])

(N) $DDpDqrDDtDttDDsqDDpsDps$.

The symbol "D", which corresponds to the symbol "/", is the only constant occurring in this axiom; all other symbols, that is lower-case letters, are propositional variables. A function of the type "$D\alpha\beta$" means the same as "if α, then it is not true that β", or "it is not true that (α and β)". Thus "D" is a proposition-forming functor of two propositional arguments; this means that in functions of the type "$D\alpha\beta$" both "α" and "β" are propositions, and "$D\alpha\beta$" is a proposition too. [4]) Dr. Sheffer has demonstrated that a function of this type can be used to define all other functions of the theory of deduction.*) Nicod's axiom, together with definitions, suffices to lay the foundations for the entire theory of deduction. [5])

If we bear in mind that the functor "D" always precedes its arguments, and that its two arguments are propositions, we can easily analyse the structure of axiom (N). We only have to realize which propositions belong to the various occurrences of "D" as their arguments. For instance, the third "D" has the proposition "q" as its first argument, and the proposition "r" as its second argument. The second "D" has the proposition "p" as its first argument, and the proposition "Dqr" as its second argument. Further analysis is made easier by the comparison of expressions (K) and (N).

Nicod's axiom is not self-evident. I shall not try to explain its content. That it is a true proposition one can verify by the zero-one verification method, assuming the following equations: [6])

$$D00 = 1, \qquad D01 = 1,$$
$$D10 = 1, \qquad D11 = 0.$$

"0" here stands for a false proposition, while "1" stands for a true proposition. By substituting in (N) 0's and 1's for the variables in any combinations we always obtain 1 after reductions performed in accord-

[1]) Cf. Leśniewski, p. 10.

[2]) Cf. Kotarbiński, p. 247 (quoted after the English translation).

[3]) I came upon the idea of a parenthesis-free notation in 1924. I used that notation for the first time in my article Łukasiewicz (1), p. 610, footnote. See also Łukasiewicz (2) pp. 7 and 38, and Kotarbiński, p. 244.

*) An English translation entitled *Elements of Mathematical Logic* is available now (published jointly in 1963 by the Polish Scientific Publishers and Pergamon Press).

[4]) The term "functor" comes from Kotarbiński. Cf. Ajdukiewicz, p. 147. The term "proposition-forming" was, as far as I know, first used by Ajdukiewicz. Cf. Ajdukiewicz, p. 16.

[5]) Historical and bibliographical information concerning the works of Sheffer and Nicod can be found in Leśniewski, pp. 9–10. Definitions of some functions, best known in propositional calculus, by means of the symbol "/" or "D" are given in Kotarbiński, p. 172, and Łukasiewicz (2), pp. 56–57.

[6]) For the zero-one verification method see Kotarbiński, pp. 159–163

*) Today, instead of "theory of deduction" we prefer the term "propositional calculus".

ance with the equations quoted above. For instance, if we put $p/0$, $q/1$, $r/0$, $s/1$, $t/0$, we obtain:

$$DD0D10DD0D00DD11DD01D01 = DD01DD01D0D11 = D1D1D00 = D1D11 = D10 = 1.$$

In deducing consequences from his axiom Nicod uses the rule of substitution and the rule of detachment.

The rule of substitution, which he does not formulate,[7]) permits us to join to the system those theses which are obtained from theses already belonging to the system by the substitution for variables of significant expressions of the system. In the system in question, every lower-case letter is a significant expression, as is the expression "$D\alpha\beta$" if both "α" and "β" are significant expressions. All significant expressions are propositions.

The rule of detachment is adopted by Nicod in the form which is equivalent to the following formulation: if a thesis of the type "$D\alpha D\beta\gamma$" belongs to the system, as does a thesis of the form of "α", then a thesis of the form of "γ" may be joined to the system. This rule becomes self-evident if we note that the expression "$D\alpha D\beta\gamma$" means the same as "if α, then it is not true that $D\beta\gamma$", and the expression "it is not true that $D\beta\gamma$" means the same as "it is not true that [it is not true that (β and γ)]", that is, "β and γ". Hence the expression "$D\alpha D\beta\gamma$" means the same as "if α, then β and γ". Hence if the whole of such an expression is asserted, and if "α" is also asserted, we may assert both "β" and "γ". But Nicod's rule of detachment disregards the expression "β" so that we may not assert that expression on the strength of that rule, nor is it necessary for us to know that it has been asserted, if we want to apply that rule.

2. The transformation which I made in Nicod's axiom consists in this, that I replaced the variable "t" by "s", thus obtaining the following thesis:

(Ł) $\quad DDpDqrDDsDssDDsqDDpsDps.$

The thesis (Ł) includes four different variables, "p", "q", "r", and "s", whereas Nicod's axiom (N) includes five, that is the four enumerated above and also the variable "t". Nevertheless, the theses (N) and (Ł)

[7]) Cf. Leśniewski, p. 10.

are equivalent, for it can be shown, by means of the rules of substitution and detachment accepted in the system, that (Ł) is a consequence of (N) and conversely (N) is a consequence of (Ł).

The proof of the first theorem, stating that (Ł) is a consequence of (N), is very easy, for it suffices to substitute "s" for "t" in (N) to obtain (Ł). The proof of the other theorem, stating that (N) is a consequence of (Ł), is not so simple and requires repeated application of the rules of substitution and detachment. That proof is recorded below by a method which must first be explained.

The starting point of the proof is the thesis (Ł), which in the proof is marked by the ordinal number "1". The terminal point of the proof is Nicod's axiom (N), which is marked by the ordinal number "11". All theses marked with ordinal numbers, except for the first, are those steps of the proof which are obtained on the strength of the rule of detachment. In the proof I note down only the theses obtained by detachment; I do not note down the theses obtained by substitution, but only mark the substitutions to be performed in order to obtain these theses.

Each thesis, except for the first, is preceded by a non-numbered line which is the "proof line" of the thesis that follows. Each proof line consists of two parts, separated from one another by a cross.*) In the first part, which precedes the cross, I mark the substitutions to be performed in some earlier thesis, already recorded in the proof. In the second part, which follows the cross, I note down the structure of the thesis obtained by means of the substitution marked before the cross, and I do it in such a way as to make it clear that the rule of detachment may be applied to that thesis. For instance, in the first part of the proof line of Thesis 2: "$1p/DpDqr$, $q/DsDss$, $r/DDsqDDpsDps$, s/t" I mark that a thesis is to be formed by substituting in 1 the expression "$DpDqr$" for "p", the expression "$DsDss$" for "q", the expression "$DDsqDDpsDps$" for "r", and the expression "t" for "s". On performing these substitutions we obtain Thesis (A), which is a step in the proof, but is not recorded in the proof in order to make the proof shorter:

*) In this paper, and in some others of his works, Łukasiewicz used an asterisk instead of a cross. For the sake of uniformity, in this volume the asterisk has been replaced everywhere by the cross.

(A) $DDDpDqrDDsDssDDsqDDpsDpsDDtDttDDtDsDssDDDpDqrtDDpDqrt$.

In the second part of this proof line: "$D1D6$-2", I mark what is the structure of Thesis (A) just formed. It begins with the letter "D", followed by an expression of the form of Thesis 1, next followed by another "D" and an expression of the form of Thesis 6, and ends with an expression of the form of Thesis 2. This shows that the rule of detachment may be applied to Thesis (A), for it is a thesis of the type "$D\alpha D\beta\gamma$", belongs to the system as a substitution of Thesis 1, and the expression which occurs in place of "α" also belongs to the system as it is of the form of Thesis 1. Thus the expression which occurs in place of "γ" may be "detached" from Thesis (A) and joined to the system as Thesis 2. An expression of the form of Thesis 6, to be obtained later on, occurs in the place of "β", but we know already that the expression "β" does not intervene in the application of the rule of detachment.

Now that the reader understands the method of writing down the proof, he can easily check all the proof lines. The best way is to take two sheets of paper, perform on one of them all the substitutions marked in the first part of a given proof line, and write out on the other the thesis occurring in the second part of that proof line. In this way the reader should obtain on both sheets identical expressions. Note that the sequence of symbols "$qr/DDpDqrt$" indicates that the expression "$DDpDqrt$" is to be substituted for both "q" and "r".

Here is the proof of the theorem stating that (N) is a consequence of (Ł):

1 $DDpDqrDDsDssDDsqDDpsDps$. (Ł)

$1\ p/DpDqr,\ q/DsDss,\ r/DDsqDDpsDps,\ s/t \times D1D6-2$.

2 $DDtDsDssDDDpDqrtDDpDqrt$.

$1\ p/DtDsDss,\ qr/DDpDqrt,\ s/w \times D2D6t/w-3$.

3 $DDwDDpDqrtDDDtDsDsswDDtDsDssw$.

$3\ w/DpDqr,\ pqr/s,\ t/DDsqDDpsDps,\ s/t \times D1D4-4$.

4 $DDDDsqDDpsDpsDtDttDpDqr$.

$2\ t/DDDstDDtsDtsDtDtt,\ s/t \times D4qpr/tD5-5$.

5 $DDpDqrDDDstDDtsDtsDtDtt$.

$5\ p/DtDsDss,\ qr/DDpDqrt \times D2D7-6$.

6 $DtDtt$.

$1\ pqr/t \times D6D6t/s-7$.

7 $DDstDDtsDts$.

$1\ p/Dst,\ qr/Dts,\ s/r \times D7D6t/r-8$.

8 $DDrDtsDDDstrDDstr$.

$7\ s/DDDsqDDpsDpsDtDtt,\ t/DpDqr \times D4D9-9$.

9 $DDpDqrDDDsqDDpsDpsDtDtt$.

$8\ r/DpDqr,\ t/DDsqDDpsDps,\ s/DtDtt \times D9D10-10$.

10 $DDDtDttDDsqDDpsDpsDpDqr$.

$7\ s/DDtDttDDsqDDpsDps,\ t/DpDqr \times D10D11-11$.

11 $DDpDqrDDtDttDDsqDDpsDps$. (N)

In this proof the rule of detachment is used 10 times, and the rule of substitution 11 times, for we have to count not only those substitutions which are marked on the left side of each of the 10 proof lines, but also the substitution "$4qpr/t$", marked in the second part of the proof line of Thesis 5. On the other hand, I do not count the substitutions "$6t/w$", "$6t/s$", "$6t/r$", marked in the proof lines of Theses 3, 7, and 8, since they pertain to those expressions which are disregarded in the detachment. Thus, in order to pass from Nicod's axiom (N) to my axiom (Ł) it is necessary to perform 21 steps of proof. I do not know how to reduce that number.[8])

The proof is complete, although it is recorded in an abbreviated manner. Moreover, the proof is formalized, which means that any one who knows the rules of inference used in the proof can verify the correctness of the proof by referring exclusively to the form of the theses and disregarding their meanings.

My axiom may be considered as a simplification of Nicod's axiom if both are noted down not by means of real, i.e., free, variables, that is if both axioms are preceded by universal quantifiers which bind the variables occurring in the axioms. On introducing an expression of the type "$\prod\alpha$", which means "for every α" and using the parenthesis-free notation of expressions with quantifiers,[9]) we obtain the following

[8]) Leśniewski, p. 10, mentions 24 steps of the proof. In fact, the proof in my manuscript of 1925, which was the basis of Dr. Leśniewski's reference, had that many steps. Now, then preparing that proof for publication I have succeeded in simplifying it by reducing the number of steps by three.

[9]) Cf. Łukasiewicz (2), pp. 78 ff.

theses:

(No) $\prod p \prod q \prod r \prod s \prod t DDpDqrDDtDttDDsqDDpsDps.$

(Ło) $\prod p \prod q \prod r \prod s DDpDqrDDsDssDDsqDDpsDps.$

In this form my axiom is shorter, and hence simpler, than Nicod's axiom.[10])

II

3. The above considerations would, perhaps, have little significance had they not revealed a certain logical fact which at first seemed paradoxical to me. Let us compare once more the axioms (N) and (Ł):

(N) $DDpDqrDDtDttDDsqDDpsDps.$

(Ł) $DDpDqrDDsDssDDsqDDpsDps.$

Both axioms are valid for any values of the variables occurring in them. But whereas in axiom (N) we may substitute for the variables "*s*" and "*t*" any propositions, either the same, i.e., of identical form, or different, in the corresponding places of axiom (Ł) we may substitute only the same propositions. This is so because only one variable "*s*" in axiom (Ł) corresponds to the different variables "*s*" and "*t*" in axiom (N). (Ł) can be obtained from (N) by the "identification" of the variables "*s*" and "*t*", that is, by the substitution of the variable "*s*" for the variable "*t*", but (N) can in no way be obtained from (Ł) by substitution alone. Axiom (N) is more general than axiom (Ł), and axiom (Ł) is a special case of axiom (N). And yet there is a deductive proof which demonstrates that the more general thesis (N) follows as a conclusion from the less general thesis (Ł) as its only premiss. I have thus encountered a previously unknown and unexpected case of generalizing deduction.

[10]) Axiom (N) and (Ł) are not organic. We call "organic" a thesis of a system, no part of which is a thesis of that system. The term "organic" was in that sense first used by Dr. Leśniewski, while the definition of an "organic" thesis comes from Mr. Wajsberg. Axioms (N) and (Ł) are not organic, since some of their parts, namely "*DtDtt*" or "*DsDss*", respectively, are theses of the system. In 1927, when he knew the result of my research presented in this paper, Mr. Wajsberg demonstrated that Nicod's axiom can be equivalently replaced by the following organic thesis:

(W) $DDpDqrDDDsrDDpsDpsDpDpq.$

This result forms part of Mr. Wajsberg's M. A. thesis, not published.

When I realized the significance of this fact I started a search for similar examples in the ordinary system of the theory of deduction. I soon found such examples among the theses which include implication only. I shall discuss below the simplest of those examples.

In the implicational system the sole primitive expression is a function of the type "$C\alpha\beta$".[11]) By the expression "$C\alpha\beta$" I mean the conditional proposition, i.e., the implication, "if α, then β". In this expression "*C*" is a proposition-forming functor of two propositional arguments. Implicational theses are noted down without parentheses in a way similar to that used above with the theses with the functor "*D*".

The rule of substitution in this system is the same as in Nicod's system. Any expression that is significant in the system may be substituted for a variable. Any lower-case letter and any expression of the type "$C\alpha\beta$", if "α" and "β" are significant expressions, is a significant expression. The rule of detachment is formulated as follows: if a thesis of the type "$C\alpha\beta$" belongs to the system, and if the thesis of the form of "α" also belongs to the system, then the thesis of the form of "β" may be joined to the system.

By means of these rules we may demonstrate the equivalence of the following two theses:[12])

1 $CqCqCrCsr,$

5 $CpCqCrCsr.$

Thesis 5 includes four different variables, while Thesis 1 includes only three such variables. Thesis 1 can be deduced from Thesis 5 by substituting in 5 the variable "*q*" for the variable "*p*". Thesis 5 can be inferred from Thesis 1 by substitution and detachment.

Here is the complete proof, noted down in an abbreviated form in a manner analogous to the proof of thesis (N) on the strength of thesis (Ł):

1 $CqCqCrCsr.$

$1\ q/CqCqCrCsr \times C1-2.$

2 $CCqCqCrCsrCrCsr.$

[11]) On the meaning of this function cf. Łukasiewicz (2), pp. 28–31. On the axioms of the implicational system see Łukasiewicz (2), p. 47 [see also the end of the present article and footnote*), p. 196 of this article].

[12]) Cf. Łukasiewicz (2), pp. 44–45, where this example is given for the first time, together with a mention about generalizing deduction.

$2 \times C1-3.$

3 $CrCsr.$

$3 r/CrCsr, s/q \times C3-4.$

4 $CqCrCsr.$

$3 r/CqCrCsr, s/p \times C4-5.$

5 $CpCqCrCsr.$

Let me add for explanation's sake that in this proof, on the left side of the first proof line, there is an indication of the substitution "1 $q/CqCqCrCsr$", the performance of which yields the following thesis, not recorded in the proof:

(T) $CCqCqCrCsrCCqCqCrCsrCrCsr.$

The structure of Thesis (T) is recorded on the right side of this proof line in the symbols "$C1$–2". This thesis begins with the letter "C", followed first by an expression of the form of Thesis 1, and next by an expression of the form of Thesis 2. This shows that the rule of detachment is applicable to Thesis (T). This is so because it is a thesis of the type "$C\alpha\beta$", it belongs to the system as a substitution of Thesis 1, and the expression which occurs in it in place of "α" also belongs to the system, since it is of the form of Thesis 1. Hence we may detach from Thesis (T) the expression which occurs in it in place of "β" by joining to the system, as Thesis 2, an expression of the form of "β". Further proof lines can easily be checked by the reader himself. The whole proof consists of seven steps, three substitutions and four detachments. I shall now analyse the proof in detail.

4. My intention is to explain first the meanings of Theses 1 and 5 and to convince the reader that they are true and in confirmity with intuition.

The proof given above shows that Thesis 3 is a consequence of Thesis 1, and Thesis 5 is a consequence of Thesis 3. Since in turn Thesis 1 is, by substitution, a consequence of Thesis 5, it follows that all three theses, 1, 3, and 5, are equivalent with one another. Let us now examine the meaning of the shortest of them, i.e., Thesis 3.

This thesis reads: "if *r*, then if *s*, then *r*". The terms "*r*" and "*s*" stand for any propositions. This thesis will not appear self-evident to everyone. And yet it can be deduced from the most self-evident theses. No one will deny that whatever propositions "*r*" and "*s*" we consider it is true that:

I. If *r* and *s*, then *r*.

Nor will anyone deny that the two formulations: "if *r* and *s*, then *t*" and "if *r*, then if *s*, then *t*", are equivalent. For instance, the following formulations are equivalent: "if a number *x* is even and divisible by 3, then it is divisible by 6" and "if a number *x* is even, then if it is divisible by 3, it is divisible by 6". Hence it follows that if we consider any propositions "*r*", "*s*", and "*t*", then it is true that:

II. If (if *r* and *s*, then *t*), then [if *r*, then (if *s*, then *t*)].

These two theorems may be written in symbols as follows:

I $CKrsr.$

II $CCKrstCrCst.$

The formula "*Krs*" stands for a conjunction of the propositions "*r*" and "*s*".[13]) By substituting in II the variable "*r*" for "*t*" and by applying the rule of detachment we obtain our Thesis 3:

II $t/r \times C1-3.$

3 $CrCsr.$

Thus Thesis 3 is a consequence of self-evident theses. Its meaning might by approximately formulated thus: if one asserts a proposition "*r*" unconditionally, then he is also authorized to assert it on a condition "*s*", so that he has the right to state: "if *s*, then *r*".

Now Thesis 3 is asserted unconditionally; hence we have the right to assert it on a condition "*q*", that is, we are authorized to state: "if *q*, then if *r*, then if *s*, then *r*". This is Thesis 4 formulated verbally.

Thesis 4 also is asserted unconditionally; hence we have the right to assert it on any condition, be it the old condition "*q*" or the new condition "*p*". In this way we obtain verbal formulations of Theses 1 and 5:

1. If *q*, then if *q*, then if *r*, then if *s*, then *r*.
5. If *p*, then if *q*, then if *r*, then if *s*, then *r*.

Thus the meanings of these theses are established. The theses are true and in agreement with intuition.

[13]) Cf. Łukasiewicz (2), p. 36.

My point now is to convince the reader beyond all doubt that the following theorems are true:

(a) The proof demonstrating that Thesis 5 is a consequence of Thesis 1 is based on Thesis 1 as its only premiss.

(b) The rules of inference used in that proof have long been known and accepted as rules of deductive inference.

(c) Conclusion 5 is more general than premiss 1.

As to (a): The completeness of the proof shows that Thesis 1 is the only premiss used in the proof. The proof has no gaps; every step of the proof is recorded or marked and is based on rules of inference specified in advance.

As to (b): The rules of inference used in the proof correspond to the rules of deductive inference already known in antiquity. All the theses considered are true for any propositions "*p*", "*q*", "*r*", and "*s*", which occur in them. Hence they are also true for certain propositions, namely conditional propositions, which we substitute in the theses. For whatever is valid for any objects of a kind, is also valid for certain objects of that kind. In applying the rule of substitution we base ourselves on the principle *dictum de omni*, which was not explicitly formulated by Aristotle, but which has always been considered the foundation of this theory of the syllogism. And the theory of the Aristotelian syllogism to this day is believed to form the nucleus of deductive logic.

In applying the rule of detachment we base our argument on the Stoic syllogism called *modus ponens*:

If α, then β,
Now α,
―――――
Hence β.

No one has ever denied that this is a mode of deductive inference.

As to (c): Thesis 5 is more general than Thesis 1, since it covers all cases covered by Thesis 1 and also cases which Thesis 1 does not cover. This will become clear when we enumerate the types of these cases:

The truth of both theses in question, like all theses in the theory of deduction, depends not on the contents of the sentences "*p*", "*q*", "*r*", and "*s*", but only on their truth or falsehood. The zero-one verification method is based precisely on that fact. If we represent a false proposition by "0", and a true proposition by "1", we obtain all the types of cases covered by Thesis 5, when in that thesis we substitute for the variables 0's and 1's in all possible combinations. The number of such combinations is 16:

*C*0*C*0*C*0*C*00	*C*1*C*1*C*0*C*00
*C*0*C*0*C*0*C*10	*C*1*C*1*C*0*C*10
*C*0*C*0*C*1*C*01	*C*1*C*1*C*1*C*01
*C*0*C*0*C*1*C*11	*C*1*C*1*C*1*C*11
*C*0*C*1*C*0*C*00	*C*1*C*0*C*0*C*00
*C*0*C*1*C*0*C*10	*C*1*C*0*C*0*C*10
*C*0*C*1*C*1*C*01	*C*1*C*0*C*1*C*01
*C*0*C*1*C*1*C*11	*C*1*C*0*C*1*C*11.

All these combinations are covered by Thesis 5; on the other hand, Thesis 1 covers only the first 8 combinations written out in the upper half, that is only those in which the term that follows the first "*C*" is equiform with the term that follows the second "*C*". Hence it is evident that Thesis 1 is a special case of Thesis 5. And yet Thesis 5, more general than Thesis 1, is a consequence of the latter on the strength of deductive inference.

I realize that this is a very particular case of generalization, since it refers to only one class of objects, namely to propositions. We infer that something is true for any propositions "*p*" and "*q*", either the same or different, on the strength of the fact that something is true for the proposition "*q*". Nevertheless this case shows that at least in the sphere of these objects, generalizing deduction is possible.

5. In textbooks on logic we often encounter the view that deduction is an inference from the general to the particular. This opinion is erroneous even in the field of traditional logic[14]) for that inference by which from the sentence "no even number is an odd number" we obtain the sentence "no odd number is an even number" is certainly deductive, since it is based on the law of conversion of general negative propositions, accepted in Aristotelian logic. Yet it may not be asserted that in that inference the relation between the premiss and the conclusion is the same as between the general and the particular. Now that we have demonstrated that in certain cases we can pass, in a deductive

[14]) Cf. Kotarbiński, p. 233

manner, from the particular to the general, the incorrectness of the above characterization of deductive inference becomes even more striking.

Together with that erroneous characterization of deduction the view that deduction does not widen our knowledge is also definitively refuted. It seems that these two opinions had their source in the conviction that the principle *dictum de omni* is the foundation of Aristotelian logic, and that Aristotelian logic exhausts deductive logic. But both these convictions are erroneous. Neither is Aristotle's theory of the syllogism based exclusively on the idea contained, although not precisely formulated, in the principle *dictum de omni*, nor does that theory cover the whole of deductive logic.[15]) Along with Aristotelian logic, which is a "logic of terms", there has for ages been Stoic logic,*) which is a "logic of propositions" and which corresponds to the present-day theory of deduction.[16])

These two logical systems are essentially different, since they are concerned with different semantic categories. No Stoic syllogism, including the law of inference called *modus ponens*. is deducible from Aristotelian logic.

As long as the principle *dictum de omni* was supposed to be the foundation of all deductive logic it was possible to think that deduction is inference from the general to the particular and that it does not widen our knowledge. But when the modern "theory of deduction" was formed, and when both the Aristotelian principle *dictum de omni* in the form of the rule of substitution, and the Stoic syllogism *modus ponens* as the rule of detachment, were applied to it, it became clear that deductive inference may be as "creative" as inductive inference, without thereby losing anything of its certainty.

I disregard here further philosophical consequences connected with these results of research in order to conclude by reverting to those problems which can be handled on the basis of mathematical logic.

[15]) Concerning the axioms on which Aristotle's theory of syllogism is based see Łukasiewicz (2), p. 87 ff. See also his *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Oxford, 1951.

[16]) Cf. Łukasiewicz (2), pp. 19 ff.

*) Stoic logic was discussed by Łukasiewicz in detail in his paper "On the History of the Logic of Propositions", (see pp. 197–219 of the present volume).

For brevity's sake, let us call "generalizing theses" those theses from which more general theses can be deduced by the rules of substitution and detachment. I am concerned here above all with the following problem, which so far I have been unable to solve: what, if any, characteristics are shared by all generalizing theses? For the sake of those who might wish to investigate this problem I quote here a number of facts which I have established.

I have verified that the following theses are equivalent to one another:

(A1) *CsCCCpqrCqr*.
(A2) *CCstCCCpqrCqr*.
(A3) *CCptCCCpqrCqr*.
(A4) *CCrtCCCpqrCqr*.

Of these, Thesis (A1) is the most general. Thesis (A2) is obtained from it by the substitution "*s*/*Cst*", and Theses (A3) and (A4) are obtained from (A2) by the respective substitutions "*s*/*p*" and "*s*/*r*". Conversely, Theses (A1) and (A2) are obtained from both (A3) and (A4) by substitution and detachment, and moreover (A1) is obtained from (A2). Thus the generalizing theses here are Theses (A2), (A3), and (A4). Note that all four of these theses are equivalent to the thesis "*CCCpqrCqr*".

Further, I have verified that the following theses are equivalent to one another:

(B1) *CtCCCpqrCCCsprr*.
(B2) *CCutCCCpqrCCCsprr*.
(B3) *CCrtCCCpqrCCCsprr*.

Here, too, Thesis (B1) is the most general. Thesis (B2) is a consequence of (B1) on the strength of the substitution "*t*/*Cut*", and (B3) is a consequence of (B2) on the strength of the substitution "*u*/*r*". Conversely, both (B1) and (B2) are consequences of (B3) on the strength of substitution and detachment; in the same way (B1) is a consequence of (B2). Thus, the generalizing theses here are (B2) and (B3). All three are equivalent to the thesis "*CCCpqrCCCsprr*".

The above examples of generalizing theses have the property in common that their consequences include a thesis of the form "*CrCsr*". This property is also shared by Thesis 1, given in Section 3 as an example of a generalizing thesis But a conclusion stating that all generalizing theses share that property would be erroneous. Here is an example to the contrary. The following theses are equivalent:

(F1) *CpCqCrCsCtr*.

(F2) *CqCqCrCsCtr*.

The generalizing thesis here is (F2). But that thesis does not have among its consequences any thesis of the form "*CrCsr*"; it has as a consequence only a thesis of the form "*CrCsCtr*". These two theses: "*CrCsr*" and "*CrCsCtr*", are independent of one another,[17]) but nevertheless they have a property in common: they make it possible to form, from any asserted thesis "α", a thesis of the type "$Cs\alpha$", where "s" is a variable that does not occur in "α". It is to be investigated whether this property is common to the generalizing theses.

All the examples of generalizing theses adduced so far, not excluding Axiom (Ł), which is a transformation of Nicod's axiom, are non-organic theses.[18]) But it would be erroneous to conclude that all generalizing theses are non-organic. In 1926, Wajsberg demonstrated that every implicational thesis that does not include negation can be deduced by substitution and detachment from the following organic thesis: *)

(W1) *CCCpqCCrstCCuCCrstCCpuCst*,

which can thus serve as the sole axiom of the implicational system.[19]) I have ascertained that (W1) has as a consequence the following more general thesis:

(W2) *CCCpqCCrstCCuCCwstCCpuCst*.

(W1) is obtained from (W2) by the substitution "w/r". Thesis (W1) is thus an example of an organic generalizing thesis. The consequences of this thesis include all implicational theses.

[17]) For the method of proving the independence of theses of the propositional calculus, see Łukasiewicz (2), pp. 109 ff.

[18]) Cf. footnote 10 above.

[19]) The result obtained by Mr. Wajsberg, as given in the present paper, was part of his M. A. thesis.

*) Cf. footnote *), p. 196

I also wish to point out that all the "sole" axioms of the implicational and the implicational-negational system known to me, whether organic or non-organic, share the property that they are either generalizing theses, like Wajsberg's axiom (W1), or are "generalized" theses, like Thesis (W2), which means that they are equivalent to some of their special cases. But it would be premature to conclude that all the sole axioms of the implicational or the implicational-negational system are either generalizing or generalized theses. I have the impression that Wajsberg's organic axiom with the primitive term "*D*" has neither of these two properties, though I have not been able to prove this fact beyond all doubt. Should it be confirmed, then it could be expected that the implicational system also includes sole axioms that have neither of these two properties.

It would be interesting to solve the problems raised above, since their solution might shed some light on generalizing deduction and thus explain on what those strange facts depend.

Added while text was in proof:

Since two years have elapsed from the completion of the present paper I wish to add here some comments and some results which I have obtained in the meantime.

As to Part I

a) Dr. Leśniewski noticed many years ago that Nicod's deduction of the thesis "*DtDtt*" from Axiom (N) contains an error. As far as I am aware, that error has not been corrected. I would not mention this fact even now were it not that 1931 saw the appearance of a comprehensive three-volume treatise by Jörgen Jörgensen, Professor of the University of Copenhagen, *A Treatise in Formal Logic* (Copenhagen–London), which, following Nicod in that respect, repeats his mistakes (cf. vol. I, p. 258, and vol. II, p. 151); in particular, Theorem (17), from which the thesis "*DtDtt*" is directly deduced, is erroneous. In drawing attention to that error I also wish to state that the deduction of Thesis 6, i.e., "*DtDtt*", from Axiom (Ł), and hence, indirectly, from Axiom (N), as given in the present paper, seems to be the first correct proof of that thesis in Nicod's system.

b) In connection with the concluding remark in the Addendum to Part I, I wish to add that in 1931 I found an organic thesis which is equivalent to Nicod's axiom and which differs from Wajsberg's thesis. The thesis in question is as follows:

(M1) *DDpDqrDDpDrpDDsqDDpsDps.*

As to Part II

c) The supposition is untenable that the generalizing theses of the implicational system all have the property that their consequences include a thesis that makes it possible to form, from any thesis "α", a thesis of the type "$Cs\alpha$", where "s" is a variable that does not occur in "α". Here is an example to the contrary:

(G1) *CsCCpCpqCrCpq.*

(G2) *CCsCstCCpCpqCrCpq.*

(G3) *CCpCpqCCpCpqCrCpq.*

All these theses are equivalent to one another, from which it follows that (G2) and (G3) are generalizing theses. Yet these theses do not have the property referred to above.

d) In connection with Wajsberg's sole axiom (W1) of the implicational system, I wish to add that in 1930 I found the following axiom of the implicational system, which is the shortest of all those known to me so far:

(Ł2) *CCCpqCrsCtCCspCrp.* *)

This axiom is equivalent to the following thesis that is a special case of it:

(Ł3) *CCCpqCrsCCtuCCspCrp.*

Wajsberg's axiom (W1) of the implicational system was published in the article: J. Łukasiewicz und A. Tarski, "Untersuchungen über den Aussagenkalkül, *Comptes rendus des séances de la Société des Sciences et des Lettres de Varsovie* 23 (1930), cl. iii. **)

*) In 1936, Łukasiewicz found a 13-letter axiom of the implicational propositional calculus. In this connection see his paper "In Defence of Logistic" in the present volume, pp. 236–249. He also discussed that axiom in a separate paper, "The Shortest Axiom of the Implicational Calculus of Propositions" (see pp. 295–305 of the present volume) where he proved that there is no sole axiom of the implicational propositional calculus consisting of less than 13 letters.

**) See "Investigations into the Sentential Calculus", pp. 131–152 of the present volume.