A NEW INTEGRATOR[1]

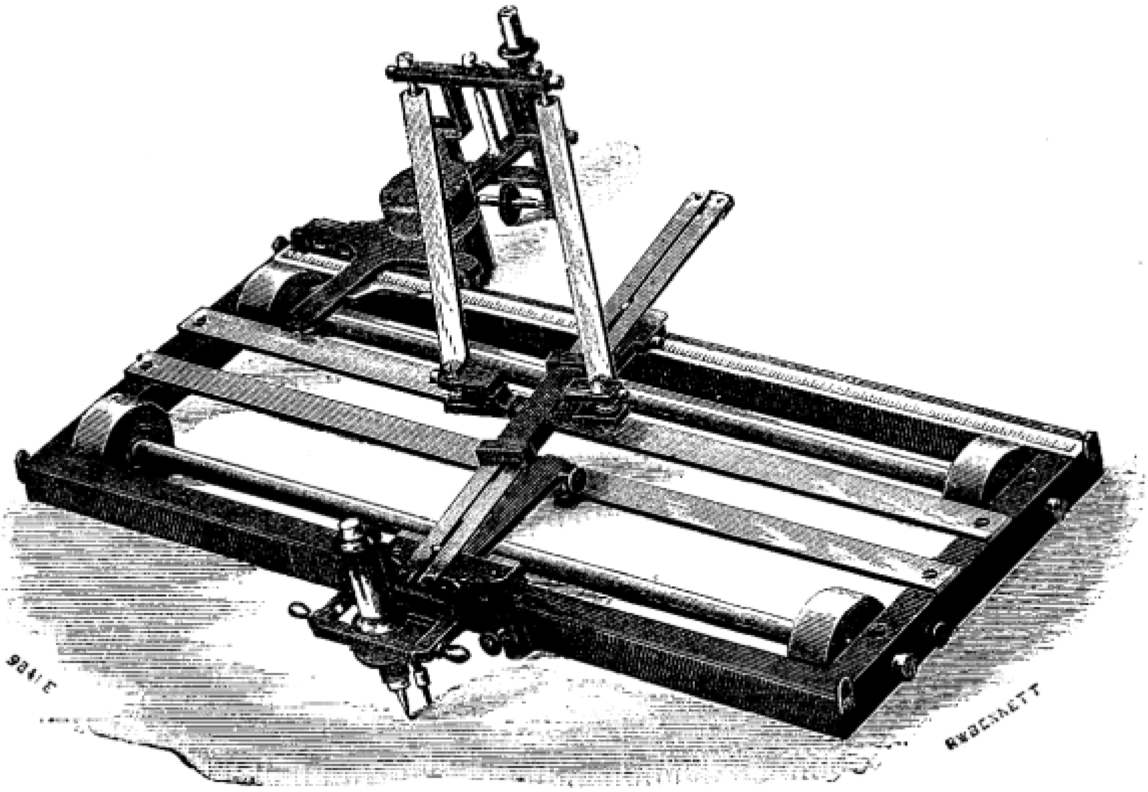
By Professor KARL PEARSON, M.A.

As I fear the title of my paper to our Society to-night contains two misstatements of fact in its three words, I must commence by correcting it. In the first place, the instrument to which I propose to draw your attention to-night is, in the narrow sense of the words, neither an integrator nor new. The name "integrator" has been especially applied to a class of instruments which measure off on a scale attached to them the magnitude of an area, arc, or other quantity. Such instruments do not, as a rule, represent their results graphically, and we may take, as characteristic examples of them, Amsler's planimeter and some of the sphere integrating machines.

An integrator which draws an absolute picture of the sum or integral is better termed an "integraph." The distinction is an important and valuable one, for while the integraph theoretically can do all the work of the integrator, the latter gives us in niggardly fashion one narrow answer, *et præterea nil*. The superiority of the integraph over the integrator cannot be better pointed out than by a concrete example. The integrator could determine by one process, the bending moment, from the shear curve, at any one chosen point of a beam; the integraph would, by an equally simple single process, gives us the bending moment at all points of the beam.

In the language of the mathematician, the integrator gives only that miserly result, a definite integral, but the integraph yields an indefinite integral, a picture of the result at all times or all points—a much greater boon in most mechanical and physical investigations. Members of our Society as students of University College have probably become acquainted with a process termed "drawing the sum curve from the primitive curve." Many have probably found this process somewhat wearisome; but this is not an unmixed evil, as the irksomeness of any manual process has more than once led to the invention of a valuable machine by the would-be idler. Thus our innate desire to take things easy is a real incentive to progress. It was some such desire as this on my part which led me, three years ago, to inquire whether a practical instrument had not been, or could not be, constructed to draw sum curves. Such an instrument is an integraph, and the one I have to describe to you to-night is the outcome of that inquiry. It is something better than my title, for it is an integraph, and not an integrator.

[1] A paper read before the University College Engineering Society on January 22.—*Engineering*.



A NEW INTEGRATOR

Before I turn to its claims to be considered new, I must first remind you of the importance of an instrument of this kind to the draughtsman. I put aside its purely mechanical applications, where it has been, or can be, attached to the indicators of steam engines, to dynamometers, dynamos, and a variety of other instruments where mechanical integration is of value. These lie entirely outside my field, and I propose only to refer to a few of the possible services of the integrator when used by hand, and not attached to a machine.

The simple finding of areas we may omit, as the planimeter will do that equally well. But of purely graphical processes which the integraph will undertake for us, I may mention the discovery of centroids, of moments of inertia (or second moments), of a scale of logarithms, of the real roots of cubic equations, and of equations of higher order (with, however, increasing labor). Further, the calculation of the cost of cutting and embanking for railways by the method of Bruckner & Culmann, the solution of a very considerable number of rather complex differential equations, various problems in the storage of water, and a great variety of statistical questions may all be completely dealt with, or very much simplified by aid of the integraph.

In graphical statics proper the integraph draws successively the curves of shear, bending moment slope, and deflection for simple beams; it does the like service for continuous beams, after certain analytical or graphical calculations have first been made; it can further lighten greatly the graphical work in the treatment of masonry arches and of metal ribs. In graphical hydrostatics it finds centers of pressure and gives a complete solution for the shear and bending moment, curves in ships, besides curves for their stability. In graphical dynamics the applications of the integraph seem still more numerous. It enables us to pass from curves of acceleration to curves of speed, and from curves of speed to curves of position. Applied to the curve of energy of either a particle or the index point of a rigid body, it enables us by the aid of easy auxiliary processes to ascertain speeds and curves of action. In a slightly altered form, that of "inverse summation," we can pass from curves of action to curves of position, and deal with a great range of resisted motions, the analysis of which still puzzles the pure mathematician; the variations of motion in flywheels, connecting rods, and innumerable other parts of mechanism, may all be calculated with much greater ease by the aid of an integraph. Shortly, it is the fundamental instrument of graphic dynamics.

It would be needless to further multiply the instances of its application; the questions we have rather to ask are: Can a practical instrument be made which will serve all these purposes? Has such an instrument been already put upon the market? If I have to answer these questions in the negative, it is rather a doubtful negative, for the instrument I have to show you to-night goes so far, and suggests so many modifications and possibilities, which would take it so much further, that it is very close to bringing the practical solution to the problem.

Let me here lay down the conditions which seem essential to a practical integraph. These are, I think, the following:

1. The price must be such that it is within the reach of the ordinary draughtsman's pocket. The Amsler's planimeter at £2 10s. or £3 may be said to satisfy this first condition. The price for the first complex integraph designed by Coradi was £24 to £30. The modified form in which I show it to-night is estimated to cost retail £14. Till an equally efficient instrument can be produced for £5 I shall not consider the price practical. If the error of its reading be not sensibly greater than that of a planimeter, it is certainly worth double the money.

2. The instrument must not be liable to get out of order by fair handling and a reasonable amount of wear and tear. I cannot speak at present with certainty as to how far our integraph satisfies this condition; it is rather too complex to quite win my confidence in this respect.

3. It must be capable of being used on the ordinary drawing board, and of having a fairly wide range on it, *i.e.*, it must not be limited to working where the primitive is at one part only of the board.

This condition takes out of every day practical drawing use the integraph invented by Professors James and Sir William Thomson, in which the sum curve is drawn on a revolving cylinder. It is essential that the sum curve should be drawn on the board not far from the primitive, and that this sum curve can be summed once or twice again without difficulty. The time involved in drawing the four sum curves, for example, required in passing from the load curve to the deflection curve of a simple beam, if these curves were drawn on different pieces of paper and had to be shifted on and off cylinders, would probably be as long as the ordinary graphical processes. Coradi's integraph works on an ordinary drawing board, but since there are nearly 10 inches between the guide point and tracer, the sum curve is thrown 10 inches behind the primitive in each integration. Thus a double summation requires say 26 inches of board, and it is impossible to integrate thrice without reproducing the primitive. The fact that the primitive and sum curve are not plotted off on the same base is also troublesome for comparison, and involves scaling of a new base for each summation. I have endeavored to obviate this by always drawing the second sum curve on a thin piece of paper pinned to the board, which can then be moved back to the position of the first primitive. But this shifting, of course, involves additional labor, and is also a source of error.

I should like to see the trace and guide chariots on the same line of rails, one below the other, were this possible without producing the bad effect of a skew, pull or push.

4. The practical integraph must not have a greater maximum error than 2 per cent. The mathematical calculations, which are correct to five or six places of decimals, are only a source of danger to the practical calculator of stresses and strains. They tend to disguise the important fact that he cannot possibly know the properties of the material within 2 per cent. error, and therefore there is not only a waste of time, but a false feeling of accuracy engendered by human and mechanical calculation which is over-refined for technical purposes.

For comparative purposes I have measured the areas of circles of 1 inch, 2 inches, and 3 inches radius, the guide being taken round the circumference by means of a "control lineal," first with an ordinary Amsler's planimeter and then with the integraph. I have obtained the following results:

			By integraph.			
Radius of circle. in.	**Calculated areas.**	**By Planimeter.**	**Middle. p=2 in.**	**Upper end. p=2 in.**	**Middle. p=4 in.**	**Upper end. p=4 in.**
1	3.14159	3.140	3.140	3.138	3.120	3.120
2	12.56636	12.55	12.36*	12.546	12.568	12.552
3	28.27431	28.24	..	..	28.280	28.288

* Cross bar had to be moved during tracing.

From this it follows that the error of the planimeter is less than 0.1 per cent. and that of the integraph about 0.5 per cent. Obviously we could make this error much less if we excluded small areas measured with large polar distances, or such polar distances that the cross bar must be shifted. Excluding such cases, we see that the accuracy of the integraph scarcely falls behind that of the planimeter and is quite efficient for practical purposes. It must be borne in mind that the above measurements were made with the "control lineal," an arrangement which carries the guide round a circle of the exact test area. In most cases the curve has to be followed by hand, and the error will be greater—greater probably for the integraph than for the planimeter, as the former is distinctly hard to guide well.

I think, then, we should be safe in saying that the error of the integraph is not likely to be greater and is probably less than 2 per cent., so that in this respect the instrument may be considered a practical one.

5. A further condition for a good integraph is that it should have a wide range of polar distances, and that it should be easily set at those distances.

One of the conditions I gave to the maker of the instrument was that it should be able to take all polar distances from one to ten half-inches. This condition he can scarcely be said to have fulfilled. With polar distances of 1/2 inch and 1 inch, the machine works unsatisfactorily, which indeed might have been foreseen from the construction of its sliding bars. It works best from 2.5 inches to 5 inches, and this is the range to which I think we ought to confine the present type of instrument. As the last conditions I may note that:

6. A practical integraph ought to be easy to read.

7. Draw a good clear curve.

The scale on the present instrument is very inconvenient, as it is often almost out of sight; the curve it draws, on the other hand, I consider very satisfactory, when the pencil is loaded, say, with a planimeter weight. On the whole, I think you will agree with me that this integraph goes a good way, if not the whole way, toward fulfilling the conditions of a practical instrument.

I next turn to its construction and the claim it has to be considered in any way new. Let me briefly remind our members of the process by which an element Q R of the sum curve (Fig. 1) corresponding to the point P on the primitive is drawn; P M being the mid-ordinate of L N, a horizontal element, P B is drawn perpendicular to any vertical line A B; and O A being a constant distance termed the base or "polar distance," Q R is drawn between the ordinates of L and W, parallel to O B. If P' be the point where P M meets Q R, we note the following relationship of P' to P.

1. If P moves along a horizontal line, O B remains unchanged, and, therefore, Q R or P' must move in the straight line Q R parallel to O B.

2. If P moves along a vertical line, P' does not change, but Q R turns round it, remaining parallel to O B.

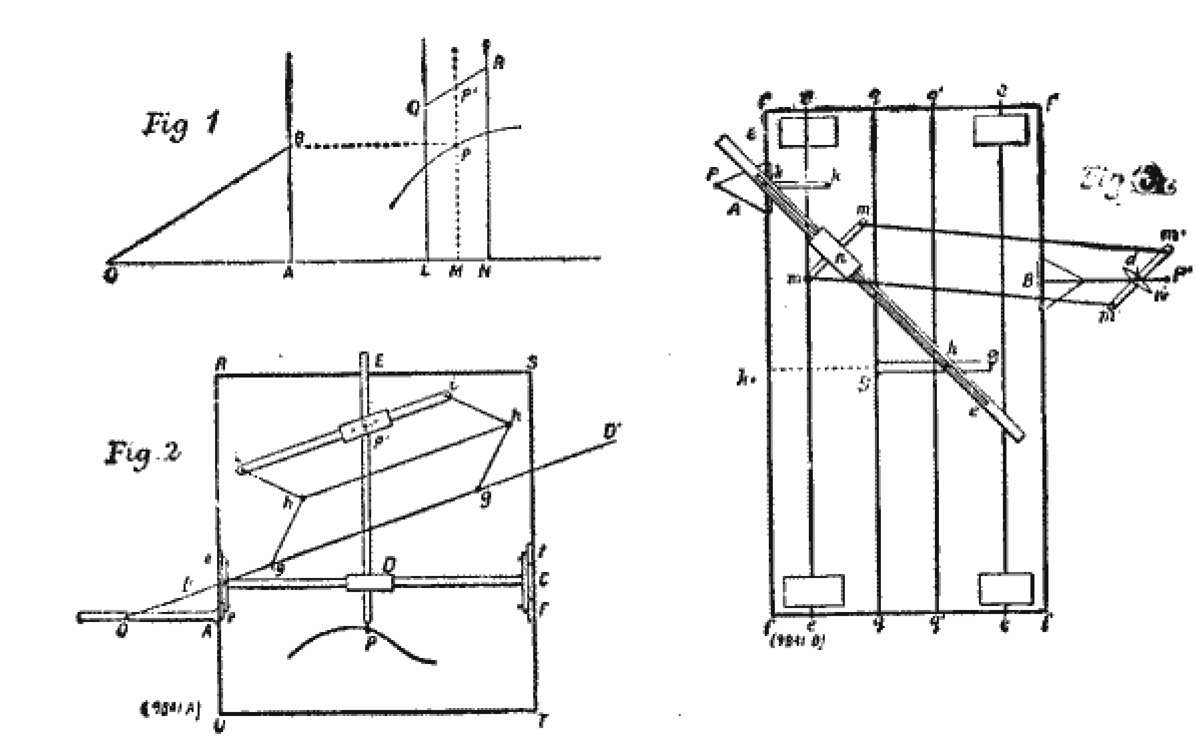


FIG. 1, 2, 3.

Without taking the trouble, as I ought to have done, to inquire what previous investigations had achieved in this matter, I thought, three years ago, I could get an apparatus to save me the trouble of drawing sum curves, made somewhat after the following fashion.

P (Fig. 2) is the guide or point to be taken round the primitive. It is attached to a block, D, which works along the bar, B C, which in its turn moves on the four wheels, e e f f, upon the frame R S U T fixed upon the drawing board. O A is fixed perpendicular to R U, and is such that O may be fixed at various points to determine the polar distance. O B D is a light bar passing freely through B and forming one side of a parallel ruler of two or more points, g g, h h, i i. Along i i is a slot and in this works a loaded block containing a wheel P', whose plane is always parallel to i i. This block also passes through a slot in D E, an arm at right angles to B C. A little consideration will show that P', if worked at all, would trace out the sum curve of P.

It was only when I showed the rough idea of this to Professor Kennedy, with the view of ascertaining what would be the amount of back-lash and friction, that I learned that Mr. Boys had already invented a very similar integrator. In his model the double parallel ruler is replaced by two endless strings and pulleys, and the bar, B C, by a T square.

Although this integrator was afterward made in a less crude form, I do not think it has ever been a practical instrument for the draughtsman. Shortly afterward I came across a work by Abdank-Abakanowicz, entitled "Les Integraphes," being a study of a "new kind of mechanical integrator."

The new kind of integrator was really only an independent version of Boys' instrument, but in many respects a great improvement. The real merit will ultimately belong to the scientific instrument maker who constructs an instrument reasonably cheap and capable of efficient practical service. Abdank-Abakanowicz's integrator however certainly went further in the practical direction than any previously constructed. The drawing board machines, it is true, of rather a complex nature, were actually exhibited to the Paris Academy, but no more have been made. The instrument before me was made by Coradi, of Zurich, on conditions laid down by me, namely, that the cost should not exceed £14, and that polar distances should range between one and ten half-inches. The first machine made by Coradi on these lines was, by a misunderstanding, sold in Germany, but the one I exhibit is the first, I believe, that has reached England, and to this extent I may, perhaps, be permitted to call it new. I look upon it rather as a suggestion upon which a still more practical instrument can be made in this country than as a perfect model. I believe there would be a wide sale for such an instrument were it once generally known to exist, and, what is more to work efficiently. It remains for me to point out in what the Abdank-Abakanowicz, or, rather, Coradi, integraph differs from Boys' instrument.

Two points deserve special attention. In the first place, the fixed frame is abolished, and the horizontal motion of P (Fig. 3), the guide point, is produced by putting the whole frame on friction rollers; in the second place, as a necessary result of the first change, the guide point carries about with it its own polar system, which renders the changes in length of "rays" much more manageable. f f, f' f' is a frame moving on four roughed wheels, e e e e, so that it can only move in the direction, f', which we may term horizontal. f f and f' f' are rails guiding the chariots, A and B, from f to f and from f' to f'. Of these chariots, A contains the guide point, P, to trace out the primitive with, and B the pencil, P', to draw the sum curve, *i.e.*, the tracer. The chariot, B, like Boys' tracer, is heavily loaded. g g is a horizontal bar rigidly attached to the crossbars, q q and q' q', of the frame. On g g is a movable pivot, to which h, which determines the pole, k_0 h being the polar distance. k_0 is the position of a second point, k, on the chariot, A, when the guide point, P, is on the initial line, g g. l l is a bar with a long slot in it, in which work the pivots, h and k; this bar represents the "ray." A projecting arm k k' has been introduced to enable me to shorten the polar distance down to 2 in. and under by removing the pivot, k to k'. m m is a bar attached to the block, n, which runs on l l, so that m m is always perpendicular to l l. On the chariot, B, is another bar, m' m', capable of turning round the pivot, d, and always maintained parallel to m m by the rods, m m', m m'. Attached to m' m' is a wheel, w, whose axis is parallel to m' m'. This wheel, therefore, always moves perpendicular to m' m', and therefore to m m; hence it moves parallel to the ray, h k. A pencil, P', attached traces out the sum curve. If we wish to use the machine as an integrator, we have merely to measure the vertical distance traversed by P', or the distance B has run along f' f'. This is done by means of a scale on f f'. If k be brought down to k_0, w runs parallel to g g, or P' traces out a horizontal straight line, which is thus the base line. If k be fixed as near as possible to k_0, which is done by means of a screw in f f at k_0, the chariot, B, can be run down f' f' as nearly opposite to k_0 as can be guessed at; a horizontal line may then be drawn as base line, and the guide point, P, brought into this line by a clamping screw with which it is provided. The instrument is then ready for action. There is a brake on one of the roughed wheels to check or stop the motion of the integraph when required.

The instrument works best when the chariots, A and B, are about opposite to each other; when they are at opposite extremities of f f and f' f' respectively, the pull at P tends to produce a skewing couple. If the chariot, B, could be put upon f f and work, if needful, by a double parallelogram from m m, we should have, excepting the skew pull, some great practical advantages. We might throw the whole of the weight of the machine on the one pair of friction wheels, and replace the other pair by a single wheel, the portion q' f' f' q' of the machine virtually disappearing. Three wheels, of course, would be a real improvement. Further, we should have the sum curve and primitive drawn to the same base line, and the simplification in the number of parts ought largely to reduce the cost of the instrument.

To be able to perform "inverse summation" (which in the language of differential calculus is to find y as a function of x, when we are given $y=f(dy/dx)$, and not $dy/dx=f(x)$ as usual), we only want a means of making the plane of the wheel, w, parallel instead of perpendicular to m' m', and it is easy to design a modification in the construction which will allow of this change.

I hope the above description of the integraph may have made its construction and method of working sufficiently clear. Those of you who have a taste for mechanical work, and the necessary tools, might, I think, with some patience, construct a workable integraph. I expect the pivots would be the hardest part of the work. I hope, some day, myself to have another instrument made with a more readily changeable polar distance, with trace and guide points working in the same vertical, and a wheel permitting of inverse summation. If this project is ever carried out, I hope I may be permitted to communicate further particulars to our society.

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